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A NEAR-RING N IN WHICH EVERY N-SUBGROUP IS AN IDEAL

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ABSTRACT

In this paper we introduce the concept of near-rings in which every N-subgroup is an ideal. In [5], S. Suryanarayanan and R. Balakrishnan investigated a near-ring N in which every N-subgroup is invariant. Motivated by this concept, we probe into the properties of a near-ring N where every N-subgroup is an ideal. We discuss the properties of this newly introduced structure, and furnish a characterization and prove that all principal ideals with mate functions is a distributive lattice.

Keywords:

near-ring
prime ideal
maximal ideal.

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1. INTRODUCTION

Near-rings are generalized rings. If in a ring (N, +,·) with two binary operations'+' and '·', we ignore the commutativity of '+' and one of the distributive laws, (N, +, ·) becomes a near-ring. If we do not stipulate the left distributive law, (N, +, ·) becomes a right near-ring. Throughout this paper, N stands for a right near-ring (N, +, ·) with at least two elements.Obviously, 0n = 0 for all n in N, where '0' denotes the identity of the group (N, +). As in [2], a subgroup (M, +) of (N, +) is called (i) a left N-subgroup of N if MN⊆M, (ii) an N-subgroup of N if NM⊆M and (iii) an invariant N – subgroup of N if Msatisfies both (i) and (ii). Again in [1], a normal subgroup (I, +) of

(N, +) is called (i) a left ideal if $n(n' + i) - nn' \in I$ for all n, $n' \in N$ and $i \in I$ (ii) a right ideal if $IN \subseteq I$ and (iii) an ideal if I satisfies both (i) and (ii). An idea I of N is called (i) a prime ideal if for all ideals J, K of N, $JK \subseteq I \Rightarrow J \subseteq I$ or $K \subseteq I$. (ii) a completely semiprime ideal if for $a \in N$, $a^2 \in I \Rightarrow a \in I$. (iii) an IFPideal, if for a, b ∈ N, $ab \in I \Rightarrow anb \in I$ for all n in N. (iv) a semiprime ideal if for all ideals J of N, $J^2 \subseteq I \Rightarrow J \subseteq I$. If {0} is a semiprime ideal, then N is called a semiprime near-ring [2.87, p.67 of Pilz [2]]. Also in [2], N is said to have property P_4 if for all ideals I of N, $ab \in I$ implies $ba \in I$ for a, b in N. The concept of a mate function in N has been introduced in [4] with a view to handling the regularity structure with considerable ease. A map '*f*' from N into N is called a mate function for N if $x=xf(x)x$ for all x in N. Also the existence of mate functions is preserved under homomorphisms. By identity 1of N, we mean only the multiplicative identity of N.
Basic concepts and terms used but left undefined in this paper can be found in Pilz [2].

2. NOTATIONS

(i) E denotes the set of all idempotents of N (e in N is called an idempotent if $e^2 = e$)
(ii) L denotes the set of all nilpotents of N (a in N is nilpotent if $a^k = 0$ for some positive integer k)
(iii) $N_d = \{n \in N / n(x+y) = nx + ny$ for all x, y in N$\}$ – set of all distributive elements of N.

(iv) $C(N) = \{n \in N / nx = xn$ for all x in N$\}$ – centre of N.

(v) $N_0 = \{n \in N / n0 = 0\}$ – zero-symmetric part of N.

3. PRELIMINARY RESULTS

We freely make use of the following results and designate them as R(1),R(2), ...etc
R(1) N has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N
R(2) If *f* is a mate function for N, then for every x in N, $xf(x), f(x)x \in E$ and $Nx = Nf(x)x$, $xN = xf(x)N$
R(3) If L={0} and $N = N_0$ then (i) $xy = 0 \Rightarrow yx = 0$ for all x, y in N (ii) N has Insertion of Factors Property– IFP for short– i.e. for x, y in N, $xy=0 \Rightarrow xny=0$ for all n in N. If N satisfies (i) and (ii) then N is said to have (∗, IFP) (Lemma 2.3 of [2])
R(4) N has strong IFP if and only if for all ideals I of N, and for x, y ∈ N, $xy \in I \Rightarrow xny \in I$ for all n ∈ N (Proposition 9.2, p.289 of [2])

R(5) N is subdirectly irreducible if and only if the intersection of any family of non-zero ideals is again nonzero (Theorem 1.60, p.25 of [2])

R(6) For any n in N, (0 : n) is a left ideal of N (1.43, p.21 of Pilz [2])

R(7) If N is zero-symmetric, then every left ideal is an N-subgroup (Proposition 1.34(b), p.19 of Pilz [2])

R(8) A zero-symmetric near-ring N has IFP if and only if (0: S) is an ideal where S is any non-empty subset of N (by 9.3, p.289 of [2])

Remark:

We denote the near-ring N in which every N-subgroup is an ideal as N.I near-ring.

Henceforth we shall in this paper choose to refer N as an N.I near-ring.

4. EXAMPLES OF NEAR-RINGS

In this section we give certain examples of this new concept.

Examples 4.1: (a) The near-ring (N, +, ·) defined on Klein's four group (N,+)with N={0,a,b,c} where '·'is defined as per scheme 22, p.408 of Pilz [2]

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	a

is a N.I near – ring.

(b) Let (N, +) be the Klein's four group as in (a) above. If multiplication is defined as per scheme 11, p.408 of Pilz [2],

·	0	a	b	c
0	0	0	0	0
a	0	a	b	a
b	0	0	0	0
c	0	a	b	a

then N is not a N.I near-ring, as the N-subgroup {0, a} is not an ideal.

5. PROPERTIES OF N.I NEAR-RINGS

In this section we prove certain important properties of N.I near-rings.

Proposition 5.1: Let N be a N.I near-ring. If N is a β_1 near-ring with identity and N = N_d, then every left N-subgroup is an ideal.

Proof: Since N is a N.I near-ring, every N-subgroup of N is an ideal. (1)

Let M be any left N-subgroup of N. Since N is a β_1 near-ring[3], M is an N-subgroup of N. This implies M is an ideal [by (1)].

Therefore, every left N-subgroup of N is an ideal.

Proposition 5.2: Let N be a N.I near-ring. Then every left N-subgroup of N is invariant.

Proof: Let M be an N-subgroup of N. Since N is a N.I near-ring, M becomes an ideal of N. Now, the desired result follows from the definition of right ideal.

Remark 5.3: The converse of Proposition 5.2 is not valid. For example, consider the near-ring (N, +,·) where (N,+) is the usual group of integers modulo 6 and where '·' is defined as per scheme 24,p.408 of Pilz [2]

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	5	5	3	1	1
2	0	4	4	0	2	2
3	3	3	3	3	3	3
4	0	2	2	0	4	4
5	3	1	1	3	5	5

We observe that, every N-subgroup of N is invariant. However N is not a N.I near-ring, since the N-subgroup {0, 3} is not an ideal.

Proposition 5.4: Let N be aN.I near-ring which admits a mate function '*f*'. Then

(i) for all N – subgroups A and B of N, A ∩ B = AB.

(ii) Nx ∩ Ny = Nxy for all x, y in N.

Proof: (i) Let A and B be two N-subgroups of N. Since N is a N.I near-ring, A and B are ideals of N. Hence $AN \subseteq A$ and $BN \subseteq B$.

Now, for $x \in A$ and $y \in B$, $xy \in AN \subseteq A$. Therefore, $AB \subseteq A$.

Also, $xy \in NB \subseteq B$. Hence $AB \subseteq B$. Consequently,

$$AB \subseteq A \cap B \qquad (1)$$

On the other hand, if $z \in A \cap B$, then since 'f' is a mate function for N, $z = zf(z)z \in (AN)B \subseteq AB$. Thus

$$A \cap B \subseteq AB \qquad (2)$$

Combining (1) and (2) $A \cap B = AB$ for all N-subgroups A, B of N.

(ii) Let $x, y \in N$. Then by taking $A = Nx$ and $B = Ny$ in (i) we get,

$$Nx \cap Ny = NxNy \qquad (3)$$

Again by taking $A = Nx$ and $B = N$ in (i) we get, $Nx = Nx \cap N = NxN$. Therefore,

$$Nxy = NxNy \qquad (4)$$

From (3) and (4), we get $Nx \cap Ny = Nxy$ for all x, y in N.

We furnish below a characterization theorem forN.Inear-rings.

Theorem 5.5: Let N be a near-ring which admits a mate function 'f'. Then the following are equivalent.

(i) N is a N.I near-ring.

(ii) Every N-subgroup is a completely semiprime ideal of N.

(iii) Every N-subgroup is an IFP ideal.

Proof:

(i) ⇒ (ii): Let M be any N-subgroup of N. Since N is a N.I near-ring, M becomes an ideal of N. Let $x^2 \in M$. Now, since 'f' is a mate function for N, $x = xf(x)x \in Nx = Nx \cap Nx = Nx^2$ [by Proposition 5.4 (ii)] $\subseteq NM \subseteq M$. Therefore, $x \in M$ and (ii) follows.

(ii)⟹ (iii): Let M be any N-subgroup of N and let $xy \in M$. Now, $(yx)(yx) = y(xy)x \in NMN \subseteq M$ by (ii). Thus we have $(yx)^2 \in M$ and (ii) implies $(yx) \in M$.For all n in N, $(x\,y)^2 = (x\,y)(x\,y) = x\,(yx)\,y \in NMN \subseteq M$ and again (ii) guarantees that $x\,y \in M$ and (iii) follows.

Remark 5.6: If N admits mate functions and is a N.I near-ring, then any homomorphic image of N also does so.

Now we are in the position to prove the main theorem for N.I near-rings.

Theorem 6.1 Let N be a N.I near-ring with a mate function and S be a multiplicatively closed subset of N not containing 0. Then any ideal P maximal with respect to P ∩S = Φ is a prime ideal.

Proof. Let ab ϵ P and suppose a ∉ P and b ∉ P. Since N is a N.I near-ring, the N-subgroups Na and N b are ideals of N. By the maximality of P, there exists s ϵ S ∩ (N a + P) and
t ϵ S ∩ (Nb + P).
Now, (Na + P)(N b + P) = NaNb + P = Nab + P [by Equation 6.5 of Proposition 5.4] ⊆ P.
Therefore, st ϵ (S ∩ (Na + P)(S ∩ (Nb + P) and this implies
st ϵ S ∩ (Na + P)(Nb + P) ϵ S ∩ P.
Hence st ϵ S ∩ P which is a contradiction. Thus P is a prime ideal.

Proposition 6.2 Let N be a N.I near-ring which admits a mate function f and let E ⊆ C(N). Then any prime ideal of N is a maximal ideal.

Proof. Let N be a N.I near-ring. Then the N-subgroups N x and N y are ideals of N. Since E ⊆ C(N), N is zero-symmetric. Further N has (∗, IF P). Let P be a prime ideal of N. Let J be an ideal of N such that J ≠ P and that P ⊆ J ⊆ N. The rest of the proof is taken care of [3]

Theorem 6.2.9 If N is a N.I near-ring with mate functions and

E ⊆ C(N), then every N-subgroup is an intersection of maximal ideals.

Proof. Since N is a N.I near-ring, every N-subgroup is an ideal Hence it suffices to show that every ideal I is an intersection of maximal ideals.
Let x ∉ I. Since f is a mate function for N, N x = N f(x)x. Let e = f(x)x ϵ E.
Then N x = Ne.
Let M be an ideal such that I ⊆ M and maximal with respect to the property
x ∉ M. (i.e.) e ∉ M.
Now, {e} is multiplicatively closed and 0 ∉ {e}, M ∩ {e} = Φ Therefore, by Theorem 6.1, M is a prime ideal. This implies M is a maximal ideal [by Proposition 6.2]. Thus we have shown that if x ∉ I, then there is a maximal ideal M containing I with x ∉ M and the proof is complete.

Lemma 6.4 Let N be a N.I near-ring with a mate function f and
$E \subseteq C(N)$. For $e_1, e_2 \in E$ there exists $e \in E$ such that
$N e_1 + N e_2 = N e$.

Proof. Since N has a mate function f, $f(x)x, xf(x) \in E$. Therefore, $E \neq \{0\}$. Also N has (*, IF P). Let $e_1, e_2 \in E$. Since N is a N.I near-ring, the N-subgroups $N e_1$, $N e_2$ becomes an ideal of N.

Now, $e_2 = e^2_2 = [e^2_2 - e_2(e_2 - e_2e_1)] + e_2(e_2 - e_2e_1)$

$\in N e_1 + N(e_2 - e_2e_1)$ [since $N e_1$ is a left ideal].
This implies

$Ne_2 \subseteq Ne_1 + N(e_2 - e_2e_1)$ and hence

$Ne_1 + Ne_2 \subseteq Ne_1 + N(e_2 - e_2e_1)$ (1)

Since Ne_1 and Ne_2 are normal subgroups of N, we have,
$Ne_1 + Ne_2 = Ne_2 + Ne_1$.

Therefore, $N(e_2 - e_2e_1) \subseteq Ne_2 + Ne_1$ implies that $N(e_2 - e_2e_1) \subseteq Ne_1 + Ne_2$.

Hence, $Ne_1 + N(e_2 - e_2e_1) \subseteq Ne_1 + Ne_2$ (2)

From Equations (1) and (2) we get

$N e_1 + N e_2 = N e_1 + N(e_2 - e_2e_1)$.

This implies that

$N e_1 + N e_2 = N e_1 + N e_3$ where $e_3 = e_2 - e_2e_1$. (3)

Since $e_3 \in Ne_3 = N(e_2 - e_2e_1)$ there exists $n \in N$ such that

$e_3 = n(e_2 - e_2e_1)$. Hence $e_3e_1 = n(e_2 - e_2e_1)e_1 = n(e_2e_1 - e_2e_1) = n.0 = 0$ [since $N = N_0$]. (i.e) $e_3e_1 = 0$ This implies $e_1e_3 = 0$. [since N has (*, IF P).

Now, we have, $(en - ene)e = 0$. This implies $e(en - ene) = 0$.
Also, $en(en - ene) = 0$ and $ene(en - ene) = 0$ [Since N has (*, IF P)]. Therefore, $en(en-ene)-ene(en-ene) = 0$ and hence $(en - ene)^2 = 0$. Since $L = \{0\}$. This demands $en = ene$ for all $e \in E$ and $n \in E$.

Therefore, $e_1(e_2 + e_3) = e_1(e_2 + e_3)e_1 = e_1$ [since $e_3e_1 = 0$] and

$e_3(e_1 + e_3) = e_3(e_1 + e_3)e_3 = e_3$ [since $e_1e_3 = 0$].

Hence $e_1 \in N(e_1 + e_3)$ and $e_3 \in N(e_1 + e_3)$.

Consequently, $N e_1 + N e_3 \subseteq N(e_1 + e_3)$ and the reverse inclusion

$N(e_1 + e_3) \subseteq N e_1 + N e_3$ is obvious.

Therefore, we get

$Ne_1 + Ne_3 = N(e_1 + e_3) = Ne$ where $e = e_1 + e_3$.

By using Equation (3), we get, $Ne_1 + Ne_2 = Ne$

And $e^2 = (e_1+e_3)^2 = (e_1+e_3)(e_1+e_3) = e_1(e_1+e_3)+e_3(e_1+e_3) = e_1 + e_3 = e$.

Thus $e \in E$ and the result follows.

Theorem 6.5 The set $\Im$ of all principal ideals of a N.I near-ring with a mate function f and $E \subseteq C(N)$ is a distributive lattice under the usual set inclusion.

Proof: We have $\Im = \{Nx/x \in N\}$

We note that $\Im = \{Ne/e=f(x)x \text{ where } x \in N\}$

For $e_1, e_2 \in N$, Proposition 5.4 (ii) guarantees that $Ne_1 \cap N e_2 = Ne_1e_2$

Further Lemma 6.4 demands that there exists $e = e_1 + e_2 - e_1e_2 \in E$

Such that $Ne_1 + Ne_2 = Ne$

The lattice axioms can be easily verified.

We need to prove the distributive axiom.

For $e_1, e_2, e_3 \in E$ we show that $Ne_1 \cap (Ne_2 + Ne_3) = Ne_1 \cap Ne_2 + Ne_1 \cap Ne_3$

Now proposition 5.4 (ii) and lemma 6.4 guarantees that

$$\begin{aligned} Ne_1 \cap (Ne_2 + Ne_3) &= Ne_1 \cap N(e_2 + e_3 - e_2e_3) \\ &= Ne_1 (e_2 + e_3 - e_2e_3) \\ &= N (e_1e_2 + e_1e_3 - e_1e_2e_3) \\ &= N (e_1e_2 + e_1e_3 - (e_1e_2) (e_1e_3)) \\ &= (N e_1e_2 + N e_1e_3) \\ &= Ne_1 \cap Ne_2 + Ne_1 \cap Ne_3 \end{aligned}$$

Hence $\Im$ is a distributive lattice under the usual set inclusion.

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